PMM U.S.S.R.,Vol.52,No.3,pp.357- 361,1988
Printed in Great Britain
0021-8928/88 $10.00+0.00

NON-LINEAR STANDING WAVES OF AN ELASTIC PLATE FLOATING ON THE SURFACE OF A HEAVY LIQUID OF INFINITE DEPTH*

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The problem of non-linear standing waves in an ideal incompressible liquid on whose surface an elastic isotropic plate floats is solved by a perturbation method. The case of bifurcation of the solution is investigated. The results of the research are discussed briefly in /1/.

1. We consider a layer of ideal incompressible liquid occupying the lower half-space, on whose surface a thin elastic isotropic plate of thickness h floats. We investigate the plane motion of the liquid, periodic in the horizontal coordinate x_1 (with period λ) and the time t_1 (with period τ). We direct the x_1 axis along the middle level of the interfacial surface between the plate and the liquid and the z_1 axis vertically upward.

The problem is to determine the velocity potential φ^*, the ordinates of the interfacial surface between the plate and the liquid ζ^* and the frequency of vibration $\sigma = 2\pi/\tau$. In dimensionless variables the problem has the form /2, 3/

$$\Delta\varphi = 0 \ (-\infty < z \leqslant \varepsilon\zeta) \tag{1.1}$$

$$z = \varepsilon\zeta, \quad \zeta_t + \varepsilon\zeta_x\varphi_x = \varphi_z \tag{1.2}$$

$$D\mu^{-1}\zeta_{xxxx} + \varkappa\zeta_{tt} + \varphi_t + {}^1/_2\varepsilon(\varphi_x^2 + \varphi_z^2) + \mu^{-1}\zeta = F(t)$$

$$z = -\infty, \quad \varphi_z = 0 \tag{1.3}$$

$$\zeta(x + 2\pi, t + 2\pi) = \zeta(x, t), \quad \int_0^{2\pi} \zeta(x, t)\,dx = 0 \tag{1.4}$$

$$x = kx_1, \quad z = kz_1, \quad t = \sigma t_1, \quad \varphi^* = \frac{\sigma}{k^2}\varepsilon\varphi, \quad \zeta^* = \frac{1}{k}\varepsilon\zeta \tag{1.5}$$

$$F^* = \frac{\varepsilon}{k^2}\sigma^2\rho F, \quad \varepsilon = ak, \quad D = \frac{Eh^3}{12(1-\nu^2)\rho g}k^4, \quad \varkappa = \frac{\rho_1}{\rho}hk,$$

$$\mu = \frac{\sigma^2}{gk}$$

Here E is the normal elastic modulus of the plate, ν is Poisson's ratio, ρ_1 is the plate density, ρ is the density of the liquid $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, a is the linear wave amplitude, and ε is a small parameter. The function $F^*(t_1)$ is also to be determined.

We will seek the solution of the problem by a perturbation method. To do this, we expand the function $\varphi(x, z, t)$ in a Taylor series in the neighbourhood of $z = 0$ and we write the non-linear boundary conditions (1.2) approximately up to terms of the order of ε^3

$$z = 0, \ \zeta_t + \varepsilon\zeta_x(\varphi_x + \varepsilon\zeta\varphi_{xz}) = \varphi_z + \varepsilon\zeta\varphi_{zz} +$$
$${}^1/_2(\varepsilon\zeta)^2\varphi_{zzz}, \quad D\mu^{-1}\zeta_{xxxx} + \varkappa\zeta_{tt} + \mu^{-1}\zeta + \varphi_t + \varepsilon\zeta\varphi_{tz} +$$
$${}^1/_2(\varepsilon\zeta)^2\varphi_{tzz} + {}^1/_2\varepsilon(\varphi_x^2 + \varphi_z^2) + \varepsilon^2\zeta(\varphi_x\varphi_{xz} + \varphi_z\varphi_{zz}) = F$$

and we represent the unknowns in the form of power series in ε:

$$\varphi = \sum_{n=0}^{\infty}\varepsilon^n\varphi_n, \quad \zeta = \sum_{n=0}^{\infty}\varepsilon^n\zeta_n, \quad \frac{1}{\mu} = \sum_{n=}^{\infty}\varepsilon^n\theta_n, \quad F = \sum_{n=0}^{\infty}\varepsilon^n F_n$$

Using the usual method (see /2/, say) and omitting the intermediate calculations, we have to third-approximation accuracy

$$\varphi^* = -\varepsilon k^{-2}\sigma_0 [\cos x \sin te^z + \varepsilon A \cos 2x \sin 2te^{2z} +$$
$$\varepsilon^2 (A_1 \cos x \sin 3te^z + A_2 \cos 3x \sin te^{3z} + A_3 \cos 3x \sin 3te^{3z})] \tag{1.6}$$

$$\zeta^* = \varepsilon k^{-1} [\cos x \cos t + \varepsilon (2B \cos 2x \cos^2 t + (C - B) \cos 2x) +$$
$$\varepsilon^2 (B_1 \cos x \cos t + B_2 \cos x \cos 3t + B_3 \cos 3x \cos t +$$
$$B_4 \cos 3x \cos 3t)] \tag{1.7}$$

*Prikl.Matem.Mekhan.,52,3,460-464,1988

$$
\begin{aligned}
&\sigma = \sigma_0\,(1 + \varepsilon^2\sigma_1) \\
&\sigma_0 = \left(\frac{gk}{\theta_0}\right)^{1/2}, \quad \sigma_1 = \frac{R}{16\,(1-R)\,(1+\varkappa)} - \frac{C}{2\,(1+\varkappa)}, \\
&F^* = -\frac{1}{2}\,\frac{\varepsilon^2}{k^2}\,\sigma_0{}^2\rho\cos 2x \\
&A = RB, \quad B = \frac{1}{4\,(1-R)}, \quad A_1 = \frac{(3R-1)\,(8\varkappa-1)+15R+39}{256\,(R-1)\,(1+\varkappa)} \\
&A_2 = \frac{3\,(3R-1)\,S-5\,(R+1)}{32\,(R-1)\,(3S-1)} - \frac{C}{2}, \quad A_3 = \frac{3(R+1)(P-1)+7R+3}{32\,(R-1)\,(P+2)} \\
&B_1 = \frac{3+R+3\varkappa\,(1+R)}{32\,(1-R)\,(1+\varkappa)} - \frac{\varkappa C}{2\,(1+\varkappa)}, \quad C = \frac{1}{4\theta_0\,(16D+1)} \\
&B_2 = \frac{4-R}{64\,(R-1)\,(1+\varkappa)}, \quad B_3 = \frac{3\,(R+3)}{16\,(1-R)\,(3S-1)}, \\
&B_4 = \frac{R+3}{16\,(1-R)\,(P+2)} \\
&R = \frac{12D_1\varkappa_1k^5+15D_1k^4-3\varkappa_1k}{1+D_1k^4}, \quad P = \frac{8\varkappa_1k-80D_1k^4-72D_1\varkappa_1k^5}{1+D_1k^4} \\
&S = \frac{80D_1\varkappa_1k^5+81D_1k^4+1}{1+D_1k^4}, \quad \theta_0 = \frac{1+\varkappa_1k}{1+D_1k^4} \\
&D_1 = D/k^4, \quad \varkappa_1 = \varkappa/k
\end{aligned}
\tag{1.8}
$$

Starting from the approximate expression for the profile (1.7) and (1.8), a number of standing-wave singularities can be established on the interfacial surface between the plate and the fluid.

1°. There are no fixed nodes. Indeed, the abscissae of the nodes are found from the equation $\zeta^*(x, t, \varepsilon) = 0$. Let us represent $\zeta^*(x, t, \varepsilon)$ in the form

$$\zeta^*(x, t, \varepsilon) = \zeta_1^*(x, t, \varepsilon) + \zeta_2^*(x, \varepsilon) \tag{1.9}$$

Since $\zeta_1^*(x, \pi/2, \varepsilon) \equiv 0$, ζ_1^* is a component of the profile being straightened out at the times $t = \pi\,(n + 1/2)$ $(n = 0, 1, 2, \ldots)$. The component ζ_2^* is idependent of the time and is a constant perturbation that agrees with the non-linear wave profile at the times mentioned. It is interesting to note that such a component appears only in the fourth approximation when there is no plate present /4/.

To determine the nodal points ζ_1^*, by confining ourselves to the first two terms of the expansion we have

$$\cos x + \varepsilon 2B\cos 2x\cos t = 0 \tag{1.10}$$

where the factor $\cos t$ has been discarded, for whose zero value only ζ_2^* remains in the full wave.

For $\varepsilon = 0$, Eq.(1.10) has the solution $x_j = \pi\,(2j-1)/2$ $(j = 1, 2)$ in the segment $[0, 2\pi]$. Let us seek the roots of (1.10) near this in the form of the series

$$x_j = \pi\,(2j-1)/2 + \varepsilon a_{1j} + \varepsilon^2 a_{2j} + \ldots \tag{1.11}$$

Substituting (1.11) into (1.10), we find to first-approximation accuracy

$$x_j = \pi\,(2j-1)/2 + (-1)^j\varepsilon\,2B\cos t \tag{1.12}$$

It follows from (1.12) that the roots of (1.10) depend on time, i.e., there are fixed nodes for the component ζ_1^*. Following /5/, we call the points under examination moving nodes, referring them to the full wave. By virtue of (1.9), we see that these nodes do not move along the horizontal but along the constant perturbation ζ_2^* near which the vibrations indeed occur.

2°. It can be shown that $\zeta_x^* = 0$ for $x = 0$ and $x = \pi$ while $\zeta_t^* = 0$ for $t = 0$ and $t = \pi$. The maximum amplitudes will be at the points: a) $x = 0$ (crest), $x = \pi$ (trough) and for $t = 0$; b) $x = 0$ (trough), $x = \pi$ (crest) and for $t = \pi$.

3°. If $B > 0$ $(B < 0)$ the amplitude is greater (less) than the amplitude of the trough, the crest is narrower (wider) and the trough is wider (narrower). This property follows from (1.7) and (1.12).

4°. Making the change of variable $t = t' + \pi/2$, we obtain that for $t' = 0$ the profile has the form

$$\zeta^* = \varepsilon^2 k^{-1}\,(C - B)\cos 2x$$

Therefore, because of the arbitrariness of the initial time the wave profile is never straightened out in the general case. However, we note that for a definite value of the wave number k' the quantity $C - B = 0$. The value of k' is the single positive root of the equation

$$\varkappa_1 - 15D_1k^3 - 14D_1\varkappa_1k^4 = 0$$

Below we present the results of calculations of the root k' for different values of the plate thickness

h, м	0.2	0.4	0.6	0.8	1.0	1.2
$10^4k'$, м$^{-1}$	368	232	177	146	126	111

Therefore, a wavelength dependent on the initial parameters of the problem exists, such that the profile of the interfacial surface between the plate and the fluid can be straightened out at certain times.

5°. It follows from (1.8) that the vibration frequency can be both greater and less than the corresponding quantity in the linear problem.

The properties 3° and 5° are associated with the fact that the equation $1-R=0$ (of fifth degree in k) has one positive real root. The equation $2+P=0$ also has a single positive root. In the neighbourhoods of these roots the amplitudes of the non-linear approximations significantly exceed the amplitude of the linear approximation. Since the amplitude of the first non-linear approximation in (1.6)-(1.8) should be at least one order of magnitude less than the amplitude of the linear approximation while the amplitude of the second non-linear term is two orders smaller, then to determine the intervals of unsuitability of the solution obtained, numerical computations were performed. The computations show that, for example, for $h=0.2$ m at least one of the quantities $|A|, |A_1-A_2+A_3|, |B+C|, |B_1+B_2+B_3+B_4|, |\sigma_1|$ is greater than four (unsuitablity of the solution for $\varepsilon \geqslant 0.1$) at each point of the interval $[0.10492; 0.10716] \cup [0.13307; 0.14023]$ (m^{-1}).

It is interesting to note that for $D_1=0$ (cracked ice) $1-R>0$, $2+P>0$, i.e., "resonance" wavelengths do not exist.

We note that the calculation of each of the subsequent approximations of the solution of the problem by a perturbation method will be accompanied by the appearance of at least one resonance wavelength, i.e., we will have a set of resonance values of the wave number, each of which will satisfy the following equation for definite m and n:

$$n(n^4-m^2)D_1\varkappa_1k^5+(n^5-m^2)D_1k^4+n(1-m^2)\varkappa_1k+n-m^2=0 \tag{1.13}$$

where m and n are simultaneously even or odd natural numbers and $m, n>1$.

For $m=n$ (1.13) has a single positive real root k_{n-1}:

$$[n(n^2+n+1)D_1]^{-1/4}<k_{n-1}<(n^2D_1)^{-1/4} \tag{1.14}$$

In the case of real parameters of the problem $D_1>0$, consequently, it follows from the estimate (1.14) that the spectrum of resonance values of the wave number obtained from (1.13) for $n=m=2, 3...$ belongs to the interval (0,1), where the zero is the condensation point. We note that this spectrum does not agree with the whole set of resonance values of the wave number. The interest in the spectrum mentioned will become comprehensible later. Below we present the results of a calculation of the roots k_1 and k_2 of the equations $1-R=0$, $2+P=0$ that yield the first two values of the spectrum

h, m	0.2	0.4	0.6	0.8	1.0	1.2
10^4k_1, m^{-1}	1368	815	602	486	411	359
10^4k_2, m^{-1}	106	632	467	377	319	278

For resonance wavelengths as well as their neighbourhoods, additional investigations must obviously be carried out.

2. We consider the case when $k=k_1$ is a root of the equation $1-R=0$. We note that this equation is equivalent to the following

$$m\sigma_0(k)=\sigma_0(nk) \tag{2.1}$$

for $m=n=2$, which is expressed in expanded form by (1.13). Here σ_0 is the linear wave vibration frequency. For $m=n$ this last equality is a special case of the synchronization condition known in non-linear optics /6/

$$\sigma_0\Big(\sum_{i=1}^{n}k_i\Big)=\sum_{i=1}^{n}\sigma_0(k_i) \tag{2.2}$$

for $n+1$ interacting waves when $k_i=k$ $(i=1, 2, \ldots, n)$.

Since the eigenvalues μ_{11} and μ_{22}, equal to one another for $k=k_1$ as follows from (2.1), correspond to the eigenfunctions

$$\varphi_{jj}=\mp\begin{Bmatrix}\sin jt\\ \cos jt\end{Bmatrix}\cos jxe^{jz}, \quad j=1, 2$$

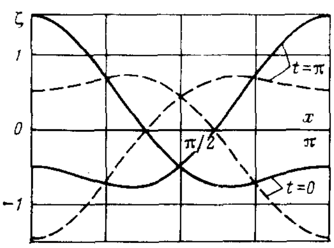
of the linear problem corresponding to (1.1)-(1.5), then the first (linear) approximation for φ is sought by the perturbation method in the form $-\cos x\sin te^z-b\cos 2x\sin 2te^{2z}$ for $k=k_1$.

Using the usual scheme of the perturbation method to find each of the approximation for φ and ζ, we obtain one undetermined coefficient which is found in calculations of the subsequent approximations. We determine the coefficient b by evaluating the second approximation. Omitting the intermediate calculations, we have to the accuracy of the second approximation

$$\begin{aligned}\zeta_\pm^* = \varepsilon k^{-1} [&\cos x \cos t + b \cos 2x \cos 2t + \varepsilon (A_1 \cos x \cos t + \\ &A_2 \cos x \cos 3t + A_3 \cos 2x \cos 2t + A_4 \cos 3x \cos t + \\ &A_5 \cos 3x \cos 3t + A_6 \cos 4x \cos 4t + A_7 \cos 4x + A_8 \cos 2x)]\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\sigma = \sqrt{\frac{gk_1}{\theta_0(k_1)}}(1+\varepsilon\sigma_1), \quad \sigma_1 = \frac{b}{4(1+\varkappa)} \\ &b = \pm\left(\frac{1+\varkappa}{4+8\varkappa}\right)^{1/2}, \quad A_1 = \frac{b(2+3\varkappa)}{4(1+\varkappa)}, \quad A_2 = -\frac{3b}{16(1+\varkappa)} \\ &A_3 = b^2\left[\frac{21+4\varkappa}{32(1+\varkappa)} + \frac{7+24\varkappa}{4(3+8\varkappa)} - \frac{3(7+12\varkappa)}{4(5+12\varkappa)} + \frac{3}{8}\right] - \\ &\quad 4b^4\left[\frac{4+9\varkappa}{7+18\varkappa} + \frac{1}{9+26\varkappa}\right] + \frac{1}{8}, \quad A_4 = \frac{b}{2(3+8\varkappa)}, \\ &A_5 = -\frac{3b}{2(5+12\varkappa)}, \quad A_6 = -\frac{b^2}{2(7+18\varkappa)}, \quad A_7 = \frac{b^2}{2(9+26\varkappa)}, \\ &A_8 = \frac{b^2}{2(1+\varkappa)}\end{aligned} \tag{2.4}$$

Therefore, for $k = k_1$ bifurcation of the solution occurs: the existence of two waves of identical length $\lambda_1 = 2\pi k_1^{-1}$, is possible, but with different vibrations frequencies, differing by an amount $2\varepsilon\sigma_0 \mid \sigma_1 \mid$. Numerical calculations show that the absolute value of the second approximation for $t = 0$, $t = \pi$ in the expression for $\zeta_\pm = \varepsilon^{-1}k_1\zeta_\pm^*$ does not exceed approximately 0.3ε, i.e., the general form of the wave profile is described by the first approximation.



The figure shows graphs of $\zeta_\pm$ at the times $t = 0$, $t = \pi$ for $\varepsilon = 0.1$ (ζ_+ is the solid line, and ζ_- the dashed line). The change in $\zeta_\pm$ as a function of the plate thickness does not exceed 0.01 when $0.2 \leqslant h \leqslant 1.2$.

3. Solutions of (1.6)-(1.8) and (2.3), (2.4) of the problem in question were obtained above, where the former is valid outside small neighbourhoods of the wave numbers k_1 and k_2 and the latter for $k = k_1$. To find the solution in the neighbourhood of $k = k_1$ we append small perturbations of the dimensionless parameters $\varkappa$ and D, which depend on k:

$$\varkappa = \varkappa_0(1 + \varepsilon d_1), \quad D = D_0(1 + \varepsilon d_2); \quad \varkappa_0 = \varkappa_1 k, \quad D_0 = D_1 k_1^4$$

To determine the profile of the interfacial surface between the plate and the fluid as well as the vibrations frequency by the perturbation method, we find expressions analogous to (2.3) and (2.4), where the coefficients in these expressions depend on $\varkappa_0$ and d_1.

Therefore, bifurcation of the solution of the problem also occurs in a small neighbourhood of $k = k_1$. Computations show that for values of k sufficiently close to k_1 the wave profiles $\zeta_\pm$ at the times $t = 0$ and $t = \pi$ are analogous in shape to the corresponding wave profiles corresponding to the solution (2.3); the shape of the profile of one wave (ζ_- for $k > k_1$, ζ_+ for $k < k_1$) tends with distance from k_1 to the shape of the wave profiles described by the solution (1.7), the amplitude of the other wave (ζ_+ for $k > k_1$ and ζ_- for $k < k_1$) grows, while the profile at the times $t = 0$ and $t = \pi$ has a definite "two-hump" shape. The properties mentioned appear most clearly for small values of the plate thickness, for instance for $h = 0.2$ m, because as h decreases the intervals comprising the intervals of unsuitability of the solution of (1.6)-(1.8) expand while the resonance values of the wave number increase.

The calculations were performed for the following parameters of the problem: $E = 3\cdot10^9$ N/m^2, $\rho = 1080$ kg/m^3, $\rho_1 = 870$ kg/m^3 and $\nu = 0.34$.

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Translated by M.D.F.

PMM U.S.S.R., Vol.52, No.3, pp.361-365, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

EXACT SOLITON SOLUTIONS OF THE GENERALIZED EVOLUTION EQUATION OF WAVE DYNAMICS*

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A Backlund transformation is proposed for the generalized evolution equation of gas dynamics, by means of which exact soliton solutions of this equation are obtained.

In recent years, a non-linear fourth-order equation has been used to describe a number of wave processes. In the general case, this takes the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\alpha\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^4 u}{\partial x^4} \tag{0.1}$$

Here α, β and γ are constant coefficients, $u(x,t)$ is a function that characterizes the physical process: mixing, the thickness of a film, concentration, etc.

With $\alpha\neq 0$, $\beta=\gamma=0$ Eq.(0.1) is the Burgers equation, which, in the simplest case, models the formation of shock waves in gas dynamics /1/. Using a Cole-Hopf transformation /2, 3/

$$u(x,t)=-2\alpha\partial\ln F/\partial x \tag{0.2}$$

the Burgers equation is transformed into a linear heat conduction equation with respect to the function $F(x,t)$. When $\alpha=\gamma=0$, $\beta\neq 0$ Eq.(0.1) is well-known as the Korteveg-de Vries (KdV) equation, which describes solitons (localized non-linear waves) /4/.

Using the Miura transformation /5, 6/

$$u(x,t)=12\beta\partial^2\ln F/\partial x^2 \tag{0.3}$$

the KdV equation reduces to an equation for $F(x,t)$ which has a quadratic form, from which Hirota /7/ found exact single- and multi-soliton solutions of the KdV equation.

Below, we will consider Eq.(0.1) with values of the coefficients α, β and γ different from zero.

1. The Backlund transformation for Eq.(0.1).

We write the solution of (0.1) in the form of the following sum:

$$u(x,t)=\sum_{j=0}^{\infty}u_j(x,t)F^{j-3}(x,t) \tag{1.1}$$

Substituting (1.1) into (0.1) and equating terms with the same powers of $F(x,t)$ we get a series of equalities:

$$\begin{gathered}u_0=-120\gamma F_x^{\,3},\ u_1=-15\beta F_x^{\,2}+180\gamma F_xF_{xx}\\ u_2=(15/76)(\beta^2/\gamma-16\alpha)F_x+15\beta F_{xx}-60\gamma F_{xxx}\end{gathered} \tag{1.2}$$

We can write the equation that contains the coefficient $u_3(x,t)$ and partial derivatives of $F(x,t)$ (denoted by F_t, F_x, F_{xx} etc.) in the form

$$\begin{gathered}F_t+u_3F_x+\frac{\beta}{76\gamma}\left(\frac{13\beta^2}{8\gamma}-7\alpha\right)F_x+\frac{15}{152}\left(\frac{\beta^2}{\gamma}-16\alpha\right)F_{xx}+\\ 5\beta F_{xxx}-15\gamma F_{xxxx}-\frac{15}{4}\beta F_{xx}^2F_x^{-1}+30\gamma F_{xx}F_{xxx}F_x^{-1}-\\ 15\gamma F_{xx}^3F_x^{-2}=0\end{gathered} \tag{1.3}$$

**Prikl.Matem.Mekhan.*, 52, 3, 465-470, 1988